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ASYMPTOTIC TRAJECTORIES AND THE STABILITY OF THE PERIODIC MOTIONS OF AN AUTONOMOUS HAMILTONIAN SYSTEM WITH TWO DEGREES OF FREEDOM*

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The existence of motions asymptotic to the periodic trajectories of a Hamiltonian system with two degrees of freedom is studied. It is assumed that the Hamiltonian function is time-independent and analytic in the neighbourhood of the periodic trajectories. It is noted that, under certain constraints, the conditions for the existence of asymptotic trajectories are equivalent to the conditions for orbital instability of the limiting periodic motion. As an application, the asymptotic trajectories in the problem of the motion of a dynamically symmetric rigid body relative to the centre of mass in a central Newtonian gravitational field in a circular orbit and in the problem of the motion of a heavy rigid body with a fixed point are considered.

1. Isoenergetic reduction. Let a generalized conservative system with two degrees of freedom have a T-periodic motion, distinct from the equilibrium position, and in the neighbourhood of the closed trajectory of the phase space corresponding to this periodic motion (PM), let the Hamiltonian function H be analytic.

Two characteristic exponents of the system of equations of the perturbed motion, linearized in the neighbourhood of the periodic motion, are always (in the case of an autonomous Hamiltonian system) equal to zero. If the other two characteristic exponents have a non-zero real part, then the PM is orbitally unstable. If they are pure imaginary (equal to $\pm i\alpha$), then both orbital instability and stability are possible, depending on the type of non-linear terms in the equations of the perturbed motion. In fact, if $k\alpha \neq n\omega$ ($\omega = 2\pi/T$; $k = 1, 2, 3, 4$; n is an integer), we usually have orbital stability; the case $k = 1, 2$ correspond to the boundary of the domains of orbital stability to a first approximation, while with $k = 3, 4$ orbital instability is possible inside these domains. A similar description of the conditions for stability and instability may be found in /1, 2/; we shall merely remark here that they are the same as the stability and instability conditions at the isoenergetic level $H = h =$ const, at which the trajectory of the PM considered lies.

To solve the problem of the existence of trajectories asymptotic to the PM trajectory, we observe that the asymptotic trajectories must correspond to the same value of the constant h as does the PM trajectory. At this fixed energy level, the equations of motion (Whittaker equations) have the form of the Hamilton equations /3/. Let us obtain these.

We can always choose /4/ (though in general this is extremely difficult) the canonical conjugate variables q_i, p_i $(i = 1, 2)$ in such a way that the PM corresponds to their values

$$q_1 = \omega t + q_{10}, \quad p_1 = q_2 = p_2 = 0 \tag{1.1}$$

where t is time, and q_{10} is the initial value of the coordinate q_1. The Hamiltonian function is then 2π-periodic in q_1.

It can be assumed without loss of generality that the trajectory of the PM (1.1) lies at the zero energy level $H = 0$. The Hamiltonian function can be expanded in a converging series

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in powers of p_1, q_2, p_2, whose coefficients are 2π-periodic in q_1. We obtain

$$H = \omega p_1 + H_2 + H_3 + H_4 + \ldots \tag{1.2}$$

where H_k is a form of degree k in p_1, q_2, p_2. The first three of these are required later; they are

$$\begin{aligned} H_2 &= h_2 + (a_1 q_2 + a_2 p_2) p_1 + a_3 p_1^2 \\ H_3 &= h_3 + (b_1 q_2^2 + b_2 q_2 p_2 + b_3 p_2^2) p_1 + (b_4 q_2 + b_5 p_2) p_1^2 + \\ &\quad b_6 p_1^3 \\ H_4 &= h_4 + f_3 p_1 + f_2 p_1^2 + f_1 p_1^3 + b_7 p_1^4 \end{aligned} \tag{1.3}$$

Here, h_k and f_l are forms of degrees k and l respectively in q_2, p_2 with coefficients that are 2π-periodic in q_1; the coefficients a_i and b_i are likewise 2π-periodic in q_1.

From the equation $H = 0$ we have

$$p_1 = -K(q_2, p_2; q_1) \tag{1.4}$$

The function K can be expanded in the convergent series

$$K = K_2 + K_3 + K_4 + \ldots \tag{1.5}$$

where K_m is a form of degree m in q_2, p_2, with coefficients 2π-periodic in q_1. For the first three forms we have

$$\begin{aligned} &K_2 = \omega^{-1} h_2, \quad K_3 = \omega^{-1} h_3 - \omega^{-2}(a_1 q_2 + a_2 p_2) h_2 \\ &K_4 = \omega^{-1} h_4 - \omega^{-2}(a_1 q_2 + a_2 p_2) h_3 - \omega^{-2}(b_1 q_2^2 + b_2 q_2 p_2 + \\ &\quad b_3 p_2^2) h_2 + \omega^{-3}(a_1 q_2 + a_2 p_2)^2 h_2 + \omega^{-3} a_3 h_2^2 \end{aligned} \tag{1.6}$$

For practical applications it is worth noting that terms of higher than the first degree in p_1 in H_3, and terms of all degrees in p_1 in H_4, do not affect terms of up to an including the fourth degree in expansion (1.5).

Whittaker equations have the form

$$dq_2/dq_1 = \partial K/\partial p_2, \quad dp_2/dq_1 = -\partial K/\partial q_2 \tag{1.7}$$

In short, in an autonomous Hamiltonian system with two degrees of freedom, the problem of the trajectories asymptotic to a closed PM trajectory reduces to the problem of the motions, asymptotic to the equilibrium position $q_2 = p_2 = 0$, of a Hamiltonian system with one degree of freedom with a 2π-periodic dependence of the Hamiltonian function $K(q_2, p_2; q_1)$ on the independent variable q_1.

If a solution $q_2 = q_2(q_1)$, $p_2 = p_2(q_1)$, of Eqs.(1.7) can be found, then p_1 can be found as a function of q_1 by substituting this solution into the right-hand side of Eq.(1.4). The dependence of q_1 on t can then be found by means of one quadrature from the equation

$$dq_1/dt = \omega + \partial(H - \omega p_1)/\partial p_1 \tag{1.8}$$

whose right-hand side is written as a function of q_1.

Given sufficiently small values of $|q_2|, |p_2|$, the function $\partial(H - \omega p_1)/\partial p_1$ can be as small as desired. In the problem of the trajectories, asymptotic to the PM trajectory, therefore, the coordinate q_1 can play the same role as the time t. To describe the asymptotic solutions of system (1.7), we shall use the results of /5, 6/.

2. Asymptotic motions in the case of real characteristic exponents. We will take the characteristic equation of the linearized system (1.7) with Hamiltonian K_2

$$\rho^2 - 2A\rho + 1 = 0 \tag{2.1}$$

In this equation, A is a constant, which can be found in the usual way /7/ from the values of the elements of the matrix of fundamental solutions of the linearized system at the point $q_1 = 2\pi$.

If $|A| > 1$, the characteristic exponents $\pm\varkappa$ of the linearized system (1.7) are real and

$$\varkappa = (2\pi)^{-1} \ln(|A| + \sqrt{A^2 - 1})$$

(Here, two characteristic exponents of the initial (unreduced) linear system of equations of the perturbed motion are also real and equal to $\pm\varkappa\omega$, while the PM is orbitally unstable). In this case /5/ the following canonical change of variables exists:

$$q_2 = \varphi(\xi, \eta; q_1), \quad p_2 = \psi(\xi, \eta; q_1) \tag{2.2}$$

given by the series φ and ψ, which converges in a sufficiently small neighbourhood of the origin $\xi = \eta = 0$ with coefficients 2π-periodic in q_1, and such that, in the new variables, the differential Eqs.(1.7) have the form

$$d\xi/dq_1 = \partial\Gamma/\partial\eta, \quad d\eta/dq_1 = -\partial\Gamma/\partial\xi \tag{2.3}$$

The analytic function Γ is independent of q_1, while ξ and η are contained in its series expansion as the product $\zeta = \xi\eta$:

$$\Gamma = \varkappa\zeta + \ldots \tag{2.4}$$

System (2.3) with Hamiltonian function (2.4) can easily be integrated. Denoting the initial values of the variables by the zero subscript, we obtain

$$\xi = \xi_0 \exp(\Gamma' q_1), \quad \eta = \eta_0 \exp(-\Gamma' q_1) \tag{2.5}$$

where Γ' is the derivative of Γ with respect to ζ, evaluated for $\zeta = \zeta_0$.

Substituting (2.5) into (2.2), we obtain the general solution of system (1.7) in a sufficiently small neighbourhood of the origin $q_2 = p_2 = 0$:

$$\begin{aligned} q_2 &= \varphi(\xi_0 \exp(\Gamma' q_1), \ \eta_0 \exp(-\Gamma' q_1); \ q_1) \\ p_2 &= \psi(\xi_0 \exp(\Gamma' q_1), \ \eta_0 \exp(-\Gamma' q_1); \ q_1) \end{aligned} \tag{2.6}$$

Solutions asymptotic to the origin as $q_1 \to +\infty$, are obtained from (2.6) with $\xi = 0$. We have

$$\begin{aligned} q_2 &= \varphi(0, \ \eta_0 \exp(-\varkappa q_1); \ q_1) \\ p_2 &= \psi(0, \ \eta_0 \exp(-\varkappa q_1); \ q_1) \end{aligned} \tag{2.7}$$

while the solutions asymptotic to the origin as $q_1 \to -\infty$, are obtained from (2.6) with $\eta_0 = 0$:

$$\begin{aligned} q_2 &= \varphi(\xi_0 \exp(\varkappa q_1), \ 0; \ q_1) \\ p_2 &= \psi(\xi_0 \exp(\varkappa q_1), \ 0; \ q_1) \end{aligned} \tag{2.8}$$

The left-hand sides of Eqs.(2.7) and (2.8) are series in powers of $\eta_0 \exp(-\varkappa q_1)$, $\xi_0 \exp(\varkappa q_1)$ respectively; the coefficients of the series are 2π-periodic functions of q_1.

To sum up, with $|A| > 1$ there are just two families of trajectories, asymptotic as $t \to \pm\infty$ to the closed PM trajectory of the initial autonomous Hamiltonian system with two degrees of freedom. The PM is itself orbitally unstable.

3. Asymptotic motions in the case of pure imaginary characteristic exponents. Now let $|A| < 1$. The characteristic exponents $\pm i\lambda$ of the linearized system (1.7) are then pure imaginary, while

$$\cos 2\pi\lambda = A \tag{3.1}$$

(In the present case, the two characteristic exponents $\pm i\alpha$ of the unreduced linear system of equations of the disturbed motion are also pure imaginary $(\alpha = \lambda\omega)$, and the PM is orbitally stable to a first approximation). If 3λ and 4λ are not integers, then the Hamiltonian function (1.5) can be reduced by means of the change of variables $q_2, p_2 \to \xi, \eta$, analytic in ξ, η, and 2π-periodic in q_1, to the form

$$\begin{aligned} &K = \lambda r + c_2 r^2 + K'(\xi, \eta; \ q_1) \\ &(\xi = \sqrt{2r}\sin\varphi, \ \eta = \sqrt{2r}\cos\varphi, \ K' = O(r^{5/2}), \ c_2 = \mathrm{const}) \end{aligned} \tag{3.2}$$

If $c_2 \neq 0$, then system (1.7) has no solutions /6/, asymptotic to the equilibrium position $q_2 = p_2 = 0$.

In the case of third-order resonances (3λ is an integer), the Hamiltonian (1.5) can be transformed /2/ to

$$K = ar^{3/2}\sin 3\varphi + O(r^2) \quad (a = \mathrm{const}) \tag{3.3}$$

If $a \neq 0$, then system (1.7) has just six /6/ families of solutions, asymptotic to the origin $q_2 = p_2 = 0$; three are asymptotic as $q_1 \to +\infty$, and the other three, as $q_1 \to -\infty$; for sufficiently large $|q_1|$, the quantities q_2 and p_2 are of the order of $|q_1|^{-1}$.

With fourth-order resonance (4λ is an integer), the Hamiltonian (1.5) can be reduced to the form

$$K = r^2(c + b\sin 4\varphi) + O(r^{5/2}) \quad (c, \ b = \mathrm{const}) \tag{3.4}$$

If $|b| < |c|$, then system (1.7) has no solutions asymptotic to the origin /6/. If $|b| > |c|$, there are just eight families of asymptotic solutions: four tend to the origin as $q_1 \to +\infty$, and four, as $q_1 \to -\infty$; for sufficiently large $|q_1|$, the quantities q_2 and p_2 are of the order of $|q_1|^{-1/2}$.

To sum up, if $|A| < 1$, and there are no third- and fourth-order resonances, and c_2 in the Hamiltonian (3.2) is non-zero, then there are no trajectories asymptotic to the PM trajectory, and the PM is orbitally stable. In the case of a third-order resonance, there are six families of asymptotic trajectories (if $a \neq 0$ in (3.3)) and the PM is orbitally unstable. In the case of fourth-order resonance, either there are eight families of asymptotic trajectories (if $|b| > |c|$), or there are no asymptotic trajectories (if $|b| < |c|$); in the former case the PM is orbitally unstable, and in the latter, orbitally stable.

4. The asymptotic trajectories of the equations of the basic problem of dynamics. The (Poincaré) basic problem of the dynamics of systems with two degrees of freedom is to investigate the trajectories for the canonical system of ordinary differential equations with the Hamiltonian function

$$H = H_0(I_1, I_2) + \mu H_1(I_1, I_2, w_1, w_2) + \ldots \tag{4.1}$$

where H is analytic in all its arguments and 2π-periodic in w_1, w_2; $0 < \mu \ll 1$.

If $\mu = 0$ the motion in the system with Hamiltonian (4.1) is given by the relations

$$I_i = I_{i0}\ (i = 1, 2), \quad w_1 = \omega_1(I_{10}, I_{20})\, t, \quad w_2 = \omega_2(I_{10}, I_{20})\, t + \sigma \tag{4.2}$$

where I_{i0} and σ are arbitrary constants, and $\omega_i = \partial H_0/\partial I_i$; it can be assumed without loss of generality that the initial value of w_1 is zero, since the equations of motion do not contain time explicitly, and the function $\omega_1(I_{10}, I_{20})$ can be assumed to be non-zero.

If the frequency ratio ω_1/ω_2 is a rational number, then the motion (4.2) is time-periodic with some period T.

If μ is non-zero but sufficiently small, the existence of T-periodic motions in the system with Hamiltonian (4.1) can be proved by Poincaré's method /8/. For this, we have to find the mean value $\langle H_1\rangle$ of the function H_1 in the unperturbed motion (4.2):

$$\langle H_1\rangle = \frac{1}{T}\int_0^T H_1(I_{10}, I_{20}, \omega_1 t, \omega_2 t + \sigma)\, dt$$

Note that $\langle H_1\rangle$ is a function of I_{10}, I_{20}, σ.

If, with $I_i = I_{i0}$, the Hessian of H_0 is non-zero and for some $\sigma = \sigma_*$ we have the conditions

$$\partial\langle H_1\rangle/\partial\sigma = 0, \quad \partial^2\langle H_1\rangle/\partial\sigma^2 \neq 0 \tag{4.3}$$

then, for sufficiently small μ, there exists a T-periodic motion which is analytic with respect to μ and transforms, for $\mu = 0$, into the motion (4.2), in which $\sigma = \sigma_*$.

Two characteristic exponents corresponding to this motion are zero, while the other two $(\pm\delta)$ can be expanded in convergent series in powers of $\sqrt{\mu}: \delta = \delta_1\sqrt{\mu} + \delta_2\mu + \ldots$, where

$$\omega_1^2\delta_1^2 = \frac{\partial^2\langle H_1\rangle}{\partial\sigma^2}\Big|_{\sigma=\sigma_*} \cdot \left(\omega_1^2\frac{\partial^2 H_0}{\partial I_2^2} - 2\omega_1\omega_2\frac{\partial^2 H_0}{\partial I_1\partial I_2} + \omega_2^2\frac{\partial^2 H_0}{\partial I_1^2}\right) \tag{4.4}$$

If the expression in parentheses in (4.4) is non-zero (i.e., the unperturbed system is isoenergetically undegenerate), then, under conditions (4.3), the number of values σ_* for which $\delta_1^2 > 0$ is equal to the number of σ_* for which $\delta_1^2 < 0$. Consequently, from the undisturbed PM (4.2) for small $\mu \neq 0$ are generated pairs of PM's; one PM of a pair is orbitally unstable (with $\delta_1^2 > 0$), and the other (with $\delta_1^2 < 0$) is orbitally stable to a first approximation.

For $\delta_1^2 > 0$ the non-zero characteristic exponents corresponding to the perturbed PM are real and of opposite sign; in accordance with Sect.2, there are in this case just two families of trajectories, asymptotic as $t \to \pm\infty$ to the trajectory of the disturbed PM.

For $\delta_1^2 < 0$ the non-zero characteristic exponents are pure imaginary. If we require additionally that, for $\sigma = \sigma_*$ the following inequality be satisfied:

$$3\frac{\partial^4\langle H_1\rangle}{\partial\sigma^4}\frac{\partial^2\langle H_1\rangle}{\partial\sigma^2} - 5\left(\frac{\partial^3\langle H_1\rangle}{\partial\sigma^3}\right)^2 \neq 0 \tag{4.5}$$

then the Poincaré PM is orbitally stable, not only to a first approximation, but also in the strict non-linear statement of the problem*. (*See: Saitbattalov A.A., Poincaré periodic solutions and their stability in the problem of the motion of a rigid body under the action of gravitational moments, Candidate Dissertation, Aviation Institute, Moscow, 1984.) In accordance with Sect.3, in this case there are no trajectories asymptotic to the PM trajectory.

5. The motions of a satellite about its centre of mass, which are asymptotic to its PM generated from plane rotations. Let the centre of mass O of the dynamically symmetric satellite (rigid body) move over a circular orbit in a central Newtonian gravitational field. The orientation of the satellite relative to the orbital coordinate system (its OX, OY, and OZ axes are directed respectively along the radius vector of the centre of mass, the velocity vector of the latter, and the binormal to the orbit) will be specified by the Euler angles ψ, θ, φ, which are introduced in the usual way. Let A and C be the equatorial and polar moments of intertia of the satellite, and ω_0 the angular velocity of the centre of mass over the orbit. As the independent variable we take the true anomaly $\nu = \omega_0 t$, while the generalized momenta, canonically conjugate with ψ, θ, φ, are reduced to dimensionless form with the aid of the factor $A\omega_0$. Since φ is a cyclical coordinate, we have $p_\varphi = \mathrm{const}$. If $p_\varphi = 0$, plane satellite motions are possible, in which its axis of symmetry always remains in the plane of the orbit.

Let $|p_\varphi|$ be small. We shall also assume that the moments of inertia A and C have close values. We put $\mu = (C - A)/(2A)$, $p_\varphi = \mu\beta$, where $|\mu| \ll 1$, $\beta = O(1)$. The Hamiltonian function corresponding to the canonical differential equations which describe the motion of the satellite about the centre of mass, has the form /9/

$$H = {}^1/_2 \sin^{-2}\theta\, p_\psi^2 - p_\psi + {}^1/_2 p_\theta^2 + \mu\,(3\sin^2\psi\cos^2\theta - \beta\cos\theta\sin^{-2}\theta p_\psi) + \ldots \tag{5.1}$$

Here and below, the dots denote a set of terms of higher than the first order in μ.

With $\mu = 0$, the equations of motion with the Hamiltonian (5.1) have the particular solution

$$\theta = {}^1/_2\pi, \quad \psi = \omega\nu + \psi_0, \quad p_\theta = 0, \; p_\psi = \sigma\,(\sigma = \omega + 1 = \mathrm{const}) \tag{5.2}$$

which corresponds (for $\omega \neq 0$) to the motion of the satellite in which its axis of symmetry rotates in the OXY plane with angular velocity $\omega\omega_0$. This motion is periodic: in a time $T = 2\pi/(|\omega|\,\omega_0)$ the axis of symmetry returns to its initial position in the orbital coordinate system.

Now let $|\mu|$ be non-zero but fairly small. Then /9/, if $1/\omega$ is not an integer, there is a satellite motion, analytic in μ, and T-periodic in t, which transforms, for $\mu = 0$, into the plane rotation (5.2). This solution can be written in the form of the series

$$\begin{aligned}
&\theta = {}^1/_2\pi - \mu\sigma^{-1}\beta + \ldots \\
&\psi = \omega\nu + \psi_0 + \mu\cdot{}^3/_4\omega^{-2}\sin 2\,(\omega\nu + \psi_0) + \ldots \\
&p_\theta = \ldots, \quad p_\psi = \sigma + \mu\cdot{}^3/_2\omega^{-1}\cos 2\,(\omega\nu + \psi_0) + \ldots
\end{aligned} \tag{5.3}$$

When studying the stability of the motion (5.3), it was shown in /9/ that on the curves in the μ, ω plane

$$\omega = -{}^3/_5 + {}^9/_4\mu + \ldots, \quad \omega = -3 - {}^9/_4\mu + \ldots \tag{5.4}$$

the PM is orbitally unstable. On the curves (5.4) there is the third-order resonance $3\lambda = -2$. Instability is also possible at resonances $3\lambda = 2l$, where l is an integer, $|l| \geqslant 2$; the curves of the resonances in the μ, ω plane issue from the points of the $\mu = 0$ axis given by the equation $\omega = 3/(2l - 3)$ (l is not a multiple of three, since otherwise $1/\omega$ would be an integer). For the remaining values of ω, for fairly small $|\mu|$, the PM (5.3) is orbitally stable. In particular, if $\omega > 3$ or $\omega < -1$, but $\omega \neq -3$, then, for small $|\mu|$, the PM (5.3) exists and is orbitally stable.

In short, in accordance with Sects.2 and 3, we can assert that, for fairly small $|\mu|$, the trajectories asymptotic to the trajectory of the PM (5.3) can exist only at third-order resonance. On the basis of the algorithm given in Sects.1 and 3, let us briefly describe the procedure for constructing the asymptotic trajectories at the resonances $3\lambda = -2$, realized with values of μ and ω which lie on curves (5.4).

After the canonical change of variables $\psi, \theta, p_\psi, p_\theta \to q_1, q_2, p_1, p_2$, given by the equations

$$\begin{aligned}
&\psi = q_1 + \mu\cdot{}^3/_4\omega^{-2}\sin 2q_1 + \ldots, \quad \theta = {}^1/_2\pi - \mu\beta\sigma^{-1} + |\sigma|^{-1/2}q_2 + \ldots \\
&p_\psi = \sigma + \mu\cdot{}^3/_2\omega^{-1}\cos 2q_1 + (1 - \mu\cdot{}^3/_2\omega^{-2}\cos 2q_1)p_1 + \ldots \\
&p_\theta = |\sigma|^{1/2}p_2 + \ldots
\end{aligned} \tag{5.5}$$

the Hamiltonian function (5.1) becomes periodic in q_1 (with period π, and not 2π, as must be the case in general, because of the structure of the Hamiltonian function in the present specific problem), while its series expansion in powers of p_1, q_2, p_2 is given by Eqs.(1.2) and (1.3), where

$$\begin{aligned}
&h_2 = {}^1/_2\,|\sigma|\,(q_2^2 + p_2^2) - \mu\cdot{}^3/_2\,|\sigma|^{-1}\,[1 - (2 + \omega^{-1})\cos 2q_1]q_2^2 + \ldots \\
&a_1 = -\mu\beta\,|\sigma|^{-1/2} + \ldots, \quad a_2 = 0 \\
&h_3 = -\mu\cdot{}^1/_2\beta\sigma^{-1}\,|\sigma|^{-1/2}\,(4 + \sigma^2 - 4\cos 2q_1)q_2^3 + \ldots
\end{aligned}$$

In the new variables, the PM (5.3) can be written as

$$p_1 = q_2 = p_2 = 0, \ q_1 = \omega\nu + \psi_0$$

After isoenergetic reduction, we obtain Eqs.(1.7), where the function K is given by Eqs. (1.5) and (1.6), in which

$$K_2 = \omega^{-1}h_2, \quad K_3 = \omega^{-1}h_3 + \mu\cdot{}^1/_2\beta \mid \sigma \mid^{1/2}\omega^{-2}q_2 (q_2^2 + p_2^2) + \ldots \tag{5.6}$$

Using a linear, π-periodic in q_1, canonical change of variables (which becomes the identity transformation when $\mu = 0$), we can /9/ introduce new variables instead of q_2, p_2 in such a way that the quadratic part of the function K takes the following form (the notation for the variables remains the same as before):

$$K_2 = {}^1/_2\lambda (q_2^2 + p_2^2) \quad (\lambda = \omega^{-1} \mid \sigma \mid - \mu\cdot{}^3/_2\cdot\omega^{-1} \mid \sigma \mid^{-1} + \ldots)$$

The function K_3 in the first-order terms in μ here remains unchanged.

Let $3\lambda = -2$, i.e., the parameters μ and ω lie on one of the curves (5.4).

Using Birkhoff's transformation /4/, we introduce new variables ξ, η in such a way that all the non-resonant terms are excluded from K_3. With $\mu = 0$ this is the identity transformation, In the new variables, the Hamiltonian K takes the form

$$K = \lambda R + aR^{3/2} \sin (3\Phi + 2q_1) + K'' (\xi, \eta, q_1, \mu) \tag{5.7}$$

$$(\xi = \sqrt{2R} \sin \Phi, \quad \eta = \sqrt{2R} \cos \Phi, \quad a = \mu\cdot{}^1/_3 \sqrt{2}\beta\sigma^{-3} \mid \sigma \mid^{-1/2})$$

The function K'' in (5.7) is a set of terms of higher than the first order of smallness in μ and higher than the third degree in ξ, η.

We again make a canonical change of variables $\Phi, R \to \varphi, r$, given by the equations

$$\Phi = \lambda q_1 + \varphi, \ R = r \tag{5.8}$$

The Hamiltonian (5.7) then becomes

$$K = ar^{3/2} \sin 3\varphi + K'' \tag{5.9}$$

If we neglect K'', we can find from the canonical equations corresponding to the Hamiltonian (5.9) the following particular solutions, corresponding to asymptotic trajectories:

$$\varphi = \varphi_k = {}^1/_3k\pi \ (k = 1, 2, \ldots, 6), \ r = r_*$$

$$r_* = 4r_0 [2 + 3a \sqrt{r_0} \cos 3\varphi_k (q_1 - q_{10})]^{-2}$$

On then returning to the initial variables q_2, p_2 (introduced by the change of variables (5.5)), we find with an error of order $\varepsilon_1 = \max (\mid \mu \mid \sqrt{r_0}, r_0)$ that

$$q_2 = \sqrt{2r_*} \sin (\lambda q_1 + \varphi_k), \ p_2 = \sqrt{2r_*} \cos (\lambda q_1 + \varphi_k) \tag{5.10}$$

If $a\omega > 0$, the solutions (5.10) with even k correspond to the trajectories, asymptotic as $t \to +\infty$ to the trajectory of PM (5.3), while the solutions (5.10) with odd k correspond to the trajectories, asymptotic to the PM (5.3) as $t \to -\infty$; if $a\omega < 0$, the reverse picture is obtained.

The value of p_1 on the asymptotic trajectories can be found from (1.4), (1.5), (5.6), and (5.10), With an error of order $\varepsilon_2 = \max (\mid \mu \mid r_0, r_0^2)$ we have

$$p_1 = {}^2/_3r_* \tag{5.11}$$

Eqs.(5.10), (5.11) and (5.5), give in the initial phase space $\psi, \theta, p_\psi, p_\theta$ the curves on which the trajectories asymptotic to the closed trajectory of the PM (5.3) lie. The coordinate q_1 plays the role of parameter on these curves. To find the dependence of q_1 on t, we have to use Eq.(1.8).

6. On the motions of a rigid body which are asymptotic to its PM and are generated from regular precessions with non-vertical axis of precession. Let the rigid body move about a fixed point in a homogeneous gravitational field. We shall assume that the principal moments of inertia of the body for the fixed point satisfy the condition $A = B \neq C$, and that the centre of gravity does not lie on the axis of symmetry and is at a small distance μd from the fixed point $(0 < \mu \ll 1, \ d = O(1))$.

With $\mu = 0$ the body performs regular prcession (we exclude the case of equilibrium of the body and of its permanent rotations about the principal axes of inertia). Let ω_1 and ω_2 be the angular velocities of the proper rotation and precession respectively, and θ_0 the angle between the axis of dynamic symmetry of the body and the kinetic moment vector. We also assume that the kinetic moment vector (which lies on the axis of precession) is non-vertical.

It has been shown /10/ that, with $A = B \neq 2C$ and $\omega_1 = \pm\omega_2$ (i.e., $\cos\theta_0 = \pm C/(A - C)$), two PM's are generated from each regular precession with small but non-zero values of μ, one of which is orbitally unstable (since there exists a pair of non-zero real characteristic exponents), and the other is orbitally stable to a first approximation (there is a pair of non-zero purely imaginary characteristic exponents).

Calculations show that condition (4.5) holds for PM's which are stable to a first approximation (the calculations are particularly simple if, as in /10/, the motion of the body is described by using the canonically conjugate Anduaille variables). Consequently, these PM's are in fact orbitally stable.

In accordance with Sect.4, for the first of these PM's there are just two families of asymptotic motions, while there are no motions asymptotic to the second PM.

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THE PARTIAL STABILITY OF MOTION*

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It is proved that the problem of stability (asymptotic stability) with respect to some of the variables, for a linear system with periodic analytic coefficients, is equivalent to the same problem with respect to all the variables, either for the same system or for an auxiliary linear system with periodic but not necessarily continuous coefficients, in less dimensions than the original system. A constructive procedure is described for constructing this auxiliary system, and the necessary and sufficient conditions are established for partial stability (asymptotic stability), generalizing the results of the Floquet-Lyapunov theory.

It is shown that the class of non-linear systems for which the problem of partial stability is solvable by linear approximation may be enlarged if, instead of the linear part of the original (non-linear) system, one considers a specially constructed linear approximating system which is equivalent to a certain non-linear subsystem of the original system. Constructive procedures are described for constructing such auxiliary systems, and a theorem on partial stability is proved. Well-known theorems on stability in the Lyapunov-critical cases are extended.

1. Formulation of the problem of the stability of a linear system with periodic coefficients. We consider a linear system of ordinary differential equations of perturbed motion:

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